Formal Distribution Algebras and Conformal Algebras

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Introduction

Conformal algebra is an axiomatic description of the operator product expansion (or rather its Fourier transform) of chiral fields in a conformal field theory. It turned out to be an adequate tool for the realization of the program of the study of Lie (super)algebras and associative algebras (and their representations), satisfying the sole locality property [K3]. The first basic definitions and results appeared in my book [K] and review [K3]. In the present paper I review recent developments in conformal algebra, including some of [K] and [K3] but in a different language. Here I use the λ-product, which is the Fourier transform of the OPE, or, equivalently, the generating series of the n-th products used in [K] and [K3]. This makes the exposition much more elegant and transparent.

Most of the work has been done jointly with my collaborators. The structure theory of finite Lie conformal algebras is a joint paper with A. D'Andrea [DK]. The theory of conformal modules has been developed with S.-J. Cheng [CK] and of their extensions with S.-J. Cheng and M. Wakimoto [CKW]. The understanding of conformal algebras Cend_N and gc_N was achieved with A. D'Andrea [DK], and of their finite representations with B. Bakalov, A. Radul and M. Wakimoto [BKRW]. The connection to Γ-local and Γ-twisted formal distribution algebras has been established with M. Golenishcheva-Kutuzova [GK] and with B. Bakalov and A. D'Andrea [BDK]. Cohomology theory has been worked out with B. Bakalov and A. Voronov [BKV].

1 Calculus of formal distributions

Let U be a vector space over $\mathbb{C}$. A U-valued *formal distribution* in one indeterminate z is a linear U-valued function on the space of Laurent polynomials $\mathbb{C}[z, z^{-1}]$. Such a formal distribution $a(z)$ can be uniquely written in the form $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$, where $a_n \in U$ is defined by $a_n = \mathrm{Res}_z z^n a(z)$ and Res_z stands for the coefficient of z^{-1}. The space of these distributions is denoted by $U[[z, z^{-1}]]$.

Likewise, one defines a U-valued formal distribution in z and w as a linear function on the space of Laurent polynomials in z and w, and such a formal distribution can be uniquely written in the form $a(z, w) = \sum_{m,n \in \mathbb{Z}} a_{m,n} z^{-m-1} w^{-n-1}$. This formal distribution defines a linear map $D_{a(z,w)} : \mathbb{C}[w, w^{-1}] \to U[[w, w^{-1}]]$ by letting $(D_{a(z,w)} f)(w) =$

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$\mathrm{Res}_z a(z,w)f(z)$. The most important $\mathbb{C}$-valued formal distribution in z and w is the *formal δ-function* $\delta(z-w)$ defined by $D_{\delta(z-w)}f = f$. Explicitly:

$$\delta(z-w) = z^{-1}\sum_{n\in\mathbb{Z}}(w/z)^n.$$

A formal distribution $a(z,w)$ is called *local* if $(z-w)^N a(z,w) = 0$ for $N \gg 0$. Note that $a(w,z)$, $\partial_z a(z,w)$ and $\partial_w a(z,w)$ are local if $a(z,w)$ is. It is easy to show ([K], Corollary 2.2) that a formal distribution $a(z,w)$ is local iff it can be represented as a finite sum of the form

$$a(z,w) = \sum_{j\in\mathbb{Z}_+} c^j(w)\partial_w^{(j)}\delta(z-w). \tag{1}$$

(Such a representation is unique.) Here and further $\partial^{(j)}$ stands for $\partial^j/j!$. This is called the *operator product expansion* (OPE), the $c^j(w)$, called the OPE coefficients, being given by

$$c^j(w) = \mathrm{Res}_z a(z,w)(z-w)^j. \tag{2}$$

Note that $a(z,w)$ is local iff $D_{a(z,w)}$ is a differential operator:
$D_{a(z,w)} = \sum_{j\in\mathbb{Z}_+} c^j(w)\partial_w^{(j)}$. Note also that $D_{a(w,z)}$ is the adjoint differential operator: $D_{a(w,z)} = \sum_{j\in\mathbb{Z}_+}(-\partial_w)^{(j)}c^j(w)$.

In order to study the properties of the expansion (1), it is convenient to introduce the *formal Fourier transform* of a formal distribution $a(z,w)$ by the formula:

$$\Phi^\lambda_{z,w}(a(z,w)) = \mathrm{Res}_z e^{\lambda(z-w)}a(z,w).$$

This is a linear map from $U[[z,z^{-1},w,w^{-1}]]$ to $U[[\lambda]][[w,w^{-1}]]$. We have:

$$\Phi^\lambda_{z,w}(\partial_w^j\delta(z-w)) = \lambda^j.$$

Hence the formal Fourier transform of the expansion (1) is

$$\Phi^\lambda_{z,w}(a(z,w)) = \sum_{n\in\mathbb{Z}_+}\lambda^{(n)}c^n(w). \tag{3}$$

(As before, $\lambda^{(n)}$ stands for $\lambda^n/n!$.) In other words, the formal Fourier transform of a formal distribution $a(z,w)$ is the generating series of its OPE coefficients. If $a(z,w)$ is local then its formal Fourier transform is polynomial in λ.

We have the following important relations:

$$\Phi^\lambda_{z,w}\partial_z = -\lambda\Phi^\lambda_{z,w} = \left[\partial_w, \Phi^\lambda_{z,w}\right], \tag{4}$$

$$\Phi^\lambda_{z,w}a(w,z) = \Phi^{-\lambda-\partial_w}_{z,w}a(z,w) \quad \text{if } a(z,w) \text{ is local.} \tag{5}$$

(The right-hand side of (5) means that the indeterminate λ in (3) is replaced by the operator $-\lambda-\partial_w$.)

Remark 1. Formulas (4) and (5) are equivalent to the following relations for the OPE coefficients $c^n_z(w)$, $c^n_w(w)$ and $\widetilde{c}^n(w)$ of the formal distributions $\partial_z a(z,w)$, $\partial_w a(z,w)$ and $a(w,z)$ respectively:

$$c^n_z(w) = -nc^{n-1}(w), \quad c^n_w(w) = \partial_w c^n(w) + nc^{n-1}(w),$$
$$\widetilde{c}^n(w) = \textstyle\sum_{j\in\mathbb{Z}_+} (-1)^{j+n} \partial_w^{(j)} c^{n+j}(w).$$

A composition of two Fourier transforms, like $\Phi^\lambda_{z,w}\Phi^\mu_{x,w}$, is a linear map from $U[[z,z^{-1},w,w^{-1},x,x^{-1}]]$ to $U[[\lambda,\mu]][[w,w^{-1}]]$. The following relation is of fundamental importance:

$$\Phi^\lambda_{z,w}\Phi^\mu_{x,w} = \Phi^{\lambda+\mu}_{x,w}\Phi^\lambda_{z,x}\,. \tag{6}$$

The proof is very easy:

$$\mathrm{Res}_z\mathrm{Res}_x e^{\lambda(z-w)+\mu(x-w)} a(z,w,x) = \mathrm{Res}_z\mathrm{Res}_x e^{\lambda(z-x)} e^{(\lambda+\mu)(x-w)} a(z,w,x).$$

2 Formal distribution algebras

Now let U be an algebra. Given two U-valued formal distributions $a(z)$ and $b(z)$, we may consider the formal distribution $a(z)b(w) = \sum_{m,n} a_m b_n z^{-m-1} w^{-n-1}$ and its formal Fourier transform, which we denote by

$$a(w)_\lambda b(w) \equiv \sum_{j\in\mathbb{Z}_+} \lambda^{(j)} (a(w)_{(j)} b(w)) = \Phi^\lambda_{z,w}(a(z)b(w)).$$

This is called the λ-*product*. The coefficients $a(w)_{(j)}b(w)$, $j\in\mathbb{Z}_+$, called j-th products, can be calculated by (2).

A pair of U-valued formal distributions $a(z)$ and $b(z)$ is called *local* if the formal distribution $a(z)b(w)$ is local (this is not symmetric in a and b in general). A subset $F\subset U[[z,z^{-1}]]$ is called a *local family* if all pairs of formal distributions from F are local. We denote by $\bar{F}$ the minimal subspace of $U[[z,z^{-1}]]$ which contains F and is closed under all j-th products.

Lemma 2 [K]**.** *Let U be a Lie (super)algebra or an associative algebra, and let $F\subset U[[z,z^{-1}]]$ be a local family. Then $\bar{F}$ is a local family as well.*

A pair (U,F), where U is an algebra and $F\subset U[[z,z^{-1}]]$, is called a *formal distribution algebra* if $\bar{F}$ is a local family of formal distributions whose coefficients span U over $\mathbb{C}$.

Let $\mathrm{Con}(U,F) = \mathbb{C}[\partial_z]\bar{F}$. This is still a local family, hence the λ-product defines a map: $\mathrm{Con}(U,F)\otimes_{\mathbb{C}}\mathrm{Con}(U,F)\to\mathbb{C}[\lambda]\otimes_{\mathbb{C}}\mathrm{Con}(U,F)$. Due to (4), the λ-product satisfies

$$\begin{aligned} \partial_z a(z)_\lambda b(z) &= -\lambda a(z)_\lambda b(z), \\ a(z)_\lambda \partial_z b(z) &= (\partial_z+\lambda)(a(z)_\lambda b(z)). \end{aligned} \tag{7}$$

Example 2. Let A be an algebra and $U = A[t,t^{-1}]$,
$F = \{a(z) = \sum_{n\in\mathbb{Z}} (at^n) z^{-n-1}\}_{a\in A}$. Since $a(z)b(w) = (ab)(w)\delta(z-w)$ for any $a,b\in A$,

we see that $F = \bar{F}$ is a local family, hence (U, F) is a formal distribution algebra. It is called a *current algebra*.

Remark 2.1. For an algebra U denote by U^{op} the algebra with the opposite multiplication $\circ$. It follows from (5) that if (U, F) is a formal distribution algebra with λ-product $a(z)_\lambda b(z)$, then (U^{op}, F) is a formal distribution algebra with λ-product

$$a(z)_\lambda \circ b(z) = b(z)_{-\lambda-\partial_z} a(z). \tag{8}$$

In the case when U is a Lie algebra we shall denote the λ-product by $[a(z)_\lambda b(z)]$ and call it the λ-*bracket*.

Proposition 2. *Let (U, F) be a formal distribution algebra.*

(a) *U is an associative algebra iff*

$$a(z)_\lambda \left(b(z)_\mu c(z)\right) = \left(a(z)_\lambda b(z)\right)_{\lambda+\mu} c(z). \tag{9}$$

(b) *U is a commutative algebra iff*

$$a(z)_\lambda b(z) = b(z)_{-\lambda-\partial_z} a(z). \tag{10}$$

(c) *U is a Lie (super)algebra iff*

$$\begin{aligned} [a(z)_\lambda b(z)] &= -p(a,b)\, [b(z)_{-\lambda-\partial_z} a(z)]\,, \\ [a(z)_\lambda\, [b(z)_\mu c(z)]] &= \Big[[a(z)_\lambda b(z)]_{\lambda+\mu}\, c(z)\Big] + p(a,b)\, [b(z)_\mu\, [a(z)_\lambda c(z)]] \end{aligned} \tag{11}$$

($p(a,b)$ stands for $(-1)^{p(a)p(b)}$, where $p(a)$ is the parity of an element $a \in U$.)

Proof. It follows from (5) and (6).

Remark 2.2. Let (U, F) be a $\mathbb{Z}_2$-graded formal distributions associative algebra where the family F is compatible with the $\mathbb{Z}_2$-gradation. Then (U_-, F), where U_- is the Lie superalgebra associated to U with the bracket $[a, b] = ab - p(a, b)ba$, is a formal distribution Lie superalgebra with the λ-bracket

$$[a(z)_\lambda b(z)] = a(z)_\lambda b(z) - p(a,b) b(z)_{-\lambda-\partial} a(z).$$

Remark 2.3. (Cf. Remark 1.) $\mathrm{Con}(U, F)$ is a $\mathbb{C}[\partial_z]$-module with a $\mathbb{C}$-bilinear product $a(z)_{(j)} b(z)$ for each $j \in \mathbb{Z}_+$ such that:

$$\begin{aligned} a(z)_{(j)} b(z) &= 0 \qquad \text{for } j \gg 0, \\ \partial_z a(z)_{(j)} b(z) &= -j a(z)_{(j-1)} b(z), \\ a(z)_{(j)} \partial_z b(z) &= \partial_z (a(z)_{(j)} b(z)) + j a(z)_{(j-1)} b(z). \end{aligned}$$

Algebra U is commutative iff

$$a(z)_{(n)}b(z) = \sum_{j\in\mathbb{Z}_+} (-1)^{j+n}\partial_z^{(j)}(b(z)_{(n+j)}a(z)).$$

Algebra U is associative iff

$$a(z)_{(m)}(b(z)_{(n)}c(z)) = \sum_{j\in\mathbb{Z}_+} \binom{m}{j} \left(a(z)_{(j)}b(z)\right)_{(m+n-j)} c(z).$$

Algebra U is a Lie superalgebra iff

$$\begin{aligned} a(z)_{(n)}b(z) &= -p(a,b)\sum_{j\in\mathbb{Z}_+} (-1)^{j+n}\partial_z^{(j)}(b(z)_{(n+j)}a(z)), \\ \left[a(z)_{(m)}, b(z)_{(n)}\right] c(z) &= \sum_{j\in\mathbb{Z}_+} \binom{m}{j} (a(z)_{(j)}b(z))_{(m+n-j)}c(z). \end{aligned}$$

3 Conformal algebras

Motivated by the discussion in the previous Section, we give the following definitions [K]:

A *conformal algebra* is a $\mathbb{C}[\partial]$-module R endowed with the λ-product $a_\lambda b$ which defines a linear map $R\otimes_{\mathbb{C}} R \to \mathbb{C}[\lambda]\otimes_{\mathbb{C}} R$ subject to the following axiom (cf. (7)):

$$(\partial a)_\lambda b = -\lambda a_\lambda b, \quad a_\lambda \partial b = (\partial+\lambda)(a_\lambda b). \tag{12}$$

We write $a_\lambda b = \sum_{j\in\mathbb{Z}_+} \lambda^{(j)}(a_{(j)}b)$ and call $a_{(j)}b$ the *j-th product* of a and b.

A conformal algebra is called *associative* if (cf. (9)):

$$a_\lambda(b_\mu c) = (a_\lambda b)_{\lambda+\mu}c, \tag{13}$$

and *commutative* if (cf. (10)):

$$a_\lambda b = b_{-\lambda-\partial}a. \tag{14}$$

The skewcommutativity and Jacobi identity read (cf. (11)):

$$\begin{aligned} [a_\lambda b] &= -p(a,b)\,[b_\lambda a]\,, \\ [a_\lambda\,[b_\mu c]] &= [[a_\lambda b]_{\lambda+\mu}c] + p(a,b)[b_\mu[a_\lambda c]]. \end{aligned} \tag{15}$$

A $\mathbb{Z}_2$-graded conformal algebra with the λ-bracket satisfying (15) is called a *Lie conformal superalgebra*. A $\mathbb{Z}_2$-graded associative conformal algebra with the λ-bracket (cf. Remark 2.2):

$$[a_\lambda b] = a_\lambda b - p(a,b)b_{-\lambda-\partial}a \tag{16}$$

is a Lie conformal superalgebra.

All these notions and formulas can be given in terms of n-th products (cf. Remark 2.3).

Remark 3. If R is a conformal algebra, then ∂R is its ideal with respect to the 0-th product $a_{(0)}b = a_\lambda b|_{\lambda=0}$.

Proposition 3 [DK]. *Any torsion element of a conformal algebra is central.*

Proof. If $P(\partial)a = 0$ for some $P(\partial) \in \mathbb{C}[\partial]$, we have:

$$0 = (P(\partial)a)_\lambda b = P(-\lambda)a_\lambda b,\ 0 = b_\lambda P(\partial)a = P(\partial+\lambda)(b_\lambda a).$$

It follows that $a_\lambda b = 0 = b_\lambda a$. □

As we have seen in the previous section, given a formal distribution algebra (U, F), one canonically associates to it a conformal algebra $\mathrm{Con}(U, F)$. An ideal I of U is called *irregular* if there exists no non-zero $b(z) \in \bar{F}$ such that all $b_n \in I$. Denote the image of F in U/I by F_1. It is clear that the canonical homomorphism $U \to U/I$ induces a surjective homomorphism $\mathrm{Con}(U, F) \to \mathrm{Con}(U/I, F_1)$, which is an isomorphism iff the ideal I is irregular.

Note that Con is a functor from the category of formal distribution algebras with morphisms being homomorphisms $\varphi : (U, F) \to (U_1, F_1)$ such that $\varphi(F) \subset \bar{F}_1$, to the category of conformal algebras with morphisms being all homomorphisms.

In order to construct the (more or less) reverse functor, we need the notion of *affinization* $\widetilde{R}$ of a conformal algebra R (which is a generalization of that for vertex algebras [B]). We let $\widetilde{R} = R[t, t^{-1}]$ with $\widetilde{\partial} = \partial + \partial_t$ and the λ-product [K]: $af(t)_\lambda bg(t) = (a_{\lambda+\partial_t}b)f(t)g(t')|_{t'=t}$. In terms of k-th products this formula reads: $at^m_{(k)}bt^n = \sum_{j\in\mathbb{Z}_+} \binom{m}{j} \left(a_{(j+k)}b\right) t^{m+n-j}$. But, by Remark 3, $\widetilde{\partial}\widetilde{R}$ is an ideal of $\widetilde{R}$ with respect to the 0-th product. We let $\mathrm{Alg}R = \widetilde{R}/\widetilde{\partial}\widetilde{R}$ with the 0-th product and let $\Re = \{\sum_{n\in\mathbb{Z}}(at^n)z^{-n-1}\}_{a\in R}$. Then $(\mathrm{Alg}R, \Re)$ is a formal distribution algebra. Note that Alg is a functor from the category of conformal algebras to the category of formal distribution algebras. One has [K]:

$$\mathrm{Con}(\mathrm{Alg}R) = R, \qquad \mathrm{Alg}(\mathrm{Con}(U, F)) = (\mathrm{Alg}\bar{F}, \bar{F}).$$

Note also that $(\mathrm{Alg}R, \Re)$ is the maximal formal distribution algebra associated to the conformal algebra R, in the sense that all formal distribution algebras (U, F) with $\mathrm{Con}(U, F) = R$ are quotients of $(\mathrm{Alg}R, \Re)$ by irregular ideals. Such formal distribution algebras are called *equivalent*.

We thus have an equivalence of categories of conformal algebras and equivalence classes of formal distribution algebras. This equivalence restricts to the categories of associative, commutative and Lie (super)algebras. So the study of formal distribution algebras reduces to the study of conformal algebras.

Example 3.1. Let A be an algebra and let $A[t, t^{-1}]$ be the associated current formal distribution algebra (see Example 2). Then the associated conformal algebra is $\mathrm{Cur}A = \mathbb{C}[\partial] \otimes_{\mathbb{C}} A$ with multiplication defined by $a_\lambda b = ab$ for $a, b \in A$ (and extended to $\mathrm{Cur}A$ by (12)). This is called the *current conformal algebra* associated to A. Note that $A[t, t^{-1}]$

is the maximal formal distribution algebra associated to Cur$A$, and that for any non-invertible Laurent polynomial $P(t)$, the formal distribution algebra $A[t,t^{-1}]/(P(t))$ is equivalent to $A[t,t^{-1}]$.

Example 3.2. The simplest formal distribution Lie algebra beyond current algebras is the Lie algebra Vect$\mathbb{C}^\times$ of regular vector fields on $\mathbb{C}^\times$ (= Lie algebra of infinitesimal conformal transformation of $\mathbb{C}^\times$, hence the choice of the term "conformal"). Vector fields $L_n = -t^{n+1}\partial_t$ ($n \in \mathbb{Z}$) form a basis of Vect $\mathbb{C}^\times$ with the familiar commutation relation $[L_m, L_n] = (m-n)L_{m+n}$. Furthermore, $L(z) = -\sum_{n\in\mathbb{Z}}(t^n\partial_t)z^{-n-1}$ is a local formal distribution (i.e. the pair (L,L) is local) since

$$[L(z), L(w)] = \partial_w L(w)\delta(z-w) + 2L(w)\delta'_w(z-w).$$

The associated Lie conformal algebra is $\mathbb{C}[\partial]L$ with λ-bracket: $[L_\lambda L] = (\partial + 2\lambda)L$. This is called the *Virasoro conformal algebra* since the centerless Virasoro algebra Vect $\mathbb{C}^\times$ is the maximal (and only) associated formal distribution Lie algebra.

A formal distribution algebra (U, F) (resp. a conformal algebra R) is called *finite* if the $\mathbb{C}[\partial]$-module $\mathbb{C}[\partial]\bar{F}$ (resp. R) is finitely generated.

4 Structure theory of finite conformal (super)algebras

A conformal algebra is called simple if it is not commutative and contains no nontrivial ideals.

Theorem 4.1 [DK]**.** *A simple finite Lie conformal algebra is isomorphic either to a current conformal algebra* Cur$\mathfrak{g}$*, where* $\mathfrak{g}$ *is a simple finite-dimensional Lie algebra, or to the Virasoro conformal algebra.*

Of course, translating this into the language of formal distribution Lie algebras, we obtain the following result:

Corollary 4. *Any formal distribution Lie algebra which is finite and simple (i.e. any its non-trivial ideal is irregular) is isomorphic either to* $(\mathrm{Vect}\mathbb{C}^\times, \{L(z)\})$ *or to a quotient of* $(\mathfrak{g}[t,t^{-1}], \{a(z)\}_{a\in\mathfrak{g}})$ *where* $\mathfrak{g}$ *is a simple finite-dimensional Lie algebra.*

Open Problem. *Classify simple formal distribution Lie algebras* $(\mathfrak{g}, F)$ *for which* F *is a finite set.*

Under the assumption that $\mathfrak{g}$ is $\mathbb{Z}$-graded the only possibilities are the Virasoro and twisted loop algebras. This follows from a very difficult theorem of Mathieu [M], but there are many examples which are not $\mathbb{Z}$-graded.

The $\mathbb{C}$-span of all elements of the form $a_{(m)}b$ of a conformal algebra R, $m \in \mathbb{Z}_+$, is called the *derived algebra* of R and is denoted by R'. It is easy to see that R' is an ideal of R such that R/R' has a trivial λ-product. We have the derived series $R \supset R' \supset R'' \supset \cdots$. A conformal algebra is called *solvable* if the n-th member of this series is zero for $n \gg 0$. A Lie conformal algebra is called *semisimple* if it contains no non-zero solvable ideals.

Theorem 4.2 [DK]**.** *Any finite semisimple Lie conformal algebra is a direct sum of conformal algebras of the following three types:*

(i) *current conformal algebra* Cur$\mathfrak{g}$, *where* $\mathfrak{g}$ *is a semisimple finite-dimensional Lie algebra,*

(ii) *Virasoro conformal algebra,*

(iii) *the semidirect product of* (i) *and* (ii), *defined by* $L_\lambda a = (\partial + \lambda)a$ *for* $a \in \mathfrak{g}$.

The proof of these results uses heavily Cartan's theory of filtered Lie algebras.

By far the hardest is the classification of finite simple Lie conformal superalgebras. The list is much richer than that of finite simple Lie conformal algebras. First, there are many more simple finite-dimensional Lie superalgebras (classified in [K1]), and the associated current conformal superalgebras are finite and simple. Second, there are many "superizations" of the Virasoro conformal algebra. They are associated to superconformal algebras constructed in [KL] and [CK1]. We describe them below.

Let $\wedge(N)$ denote the Grassmann algebra in N indeterminates $\xi_1, \dots, \xi_N$ and let $W(N)$ be the Lie superalgebra of all derivations of the superalgebra $\wedge(N)$. It consists of all linear differential operators $\sum_i P_i \partial_i$, where ∂_i stands for the partial derivative by ξ_i.

The first series of examples is the series of Lie conformal superalgebra W_N of rank $(N+1)2^N$ over $\mathbb{C}[\partial] : W_N = \mathbb{C}[\partial] \otimes_{\mathbb{C}} (W(N) + \wedge(N))$ with the following λ-brackets $(a, b \in W(N);\ f, g \in \wedge(N))$:

$$[a_\lambda b] = [a, b], \quad [a_\lambda f] = a(f) - p(a, f)\lambda f a, \quad [f_\lambda g] = -(\partial + 2\lambda) f g.$$

The second series is S_N of rank $N2^N$ over $\mathbb{C}[\partial] : S_N = \{D \in W_N | \mathrm{div} D = 0\}$, where

$$\mathrm{div}(\sum_i P_i(\partial, \xi)\partial_i + f(\partial, \xi)) = \sum_i (-1)^{p(P_i)} \partial_i P_i - \partial f.$$

The third series is a deformation of S_N (N even): $\tilde{S}_N = \{D \in W_N | \mathrm{div}(1 + \xi_1 \dots \xi_N) D = 0\}$. The fourth series is K_N, which is also a subalgebra of W_N (of rank 2^N), but it is more convenient to describe it as follows: $K_N = \mathbb{C}[\partial] \otimes_{\mathbb{C}} \wedge(N)$ with the following λ-bracket $(f, g \in \wedge(N))$:

$$[f_\lambda g] = \left(\frac{1}{2}|f| - 1\right) \partial(fg) + \frac{1}{2}(-1)^{|f|} \sum_i (\partial_i f)(\partial_i g) + \lambda \left(\frac{1}{2}|f| + \frac{1}{2}|g| - 2\right) fg.$$

We assume here that f and g are homogeneous elements of degree $|f|$ and $|g|$ in the $\mathbb{Z}$-gradation $\deg \xi_i = 1$ for all i.

Theorem 4.3 [K2], [K4]**.** *Any simple finite Lie conformal superalgebra is isomorphic either to a current conformal superalgebra* Cur$\mathfrak{g}$, *where* $\mathfrak{g}$ *is a simple finite-dimensional Lie superalgebra, or to one of the conformal superalgebras* W_N, S_{N+2}, $\tilde{S}_{2N+2}$, K_N ($N \in \mathbb{Z}_+$), *or to the exceptional conformal superalgebra* CK_6 *of rank 32 (which is a subalgebra of* K_6*) constructed in [CK1].*

The structure theory of finite associative conformal algebras is much simpler than that of Lie conformal algebras. Define the central series $R \supset R^1 \supset R^2 \supset \cdots$ by letting $R^1 = R'$ and $R^n = \mathbb{C}$-span of all products $a_{(j)}b$ and $b_{(j)}a$, $j \in \mathbb{Z}_+$, where $a \in R$, $b \in R^{n-1}$, for $n \geq 2$. A conformal algebra is called *nilpotent* if $R^n = 0$ for $n \gg 0$. An associative conformal algebra is called semisimple if it contains no non-zero nilpotent ideals.

Theorem 4.4. *Any finite semisimple associative conformal algebra is a direct sum of associative conformal algebras of the form* $\mathrm{Cur}(\mathrm{Mat}_N)$, $N \geq 1$, *where* Mat_N *stands for the associative algebra of all complex* $N \times N$ *matrices.*

5 Conformal modules and modules over conformal algebras

Now let A be an associative algebra or a Lie (super)algebra, and let V be an A-module. Given an A-valued formal distribution $a(z)$ and a V-valued formal distribution $v(z)$ we may consider the formal distribution $a(z)v(w)$ and its formal Fourier transform $a(w)_\lambda v(w) = \Phi^\lambda_{z,w}(a(z)v(w))$. This is called the λ-*action* of A on V. It has properties similar to (7):

$$\begin{aligned} \partial_z a(z)_\lambda v(z) &= -\lambda a(z)_\lambda v(z), \\ a(z)_\lambda \partial_z v(z) &= (\partial_z + \lambda)(a(z)_\lambda b(z)). \end{aligned} \tag{17}$$

In the case when A is associative we have a formula similar to (9):

$$a(z)_\lambda (b(z)_\mu v(z)) = (a(z)_\lambda b(z))_{\lambda+\mu} v(z). \tag{18}$$

Likewise, in the case when A is a Lie (super)algebra, we have a formula similar to (11b):

$$[a(z)_\lambda, b(z)_\mu] v(z) = [a(z)_\lambda b(z)]_{\lambda+\mu} v(z). \tag{19}$$

As before, the pair $(a(z),\ v(z))$ is called *local* if the formal distribution $a(z)v(w)$ is local.

Lemma 5 [K3]**.** *Let $F \subset A[[z,z^{-1}]]$ be a local family and let $E \subset V[[z,z^{-1}]]$ be such that all pairs $(a(z), v(z))$, where $a(z) \in F$ and $v(z) \in E$, are local. Let $\bar{E}$ be the minimal subspace of $V[[z,z^{-1}]]$ which contains E and all $a(z)_{(j)}v(z)$ for $a(z) \in F$, $v(z) \in \bar{E}$. Then all pairs $(a(z), v(z))$ with $a(z) \in \bar{F}$ and $v(z) \in \bar{E}$ are local. Moreover $a(z)_{(j)}(\mathbb{C}[\partial_z]\bar{E}) \subset \mathbb{C}[\partial_z]\bar{E}$ for all $a(z) \in \mathbb{C}[\partial_z]\bar{F}$.*

Let F be a local family that spans A and let $E \subset V[[z,z^{-1}]]$ be a family that spans V. Then (V, E) is called a *conformal module* over the formal distribution algebra (A, F) if all pairs $(a(z), v(z))$, where $a(z) \in F$ and $v(z) \in E$ are local. It follows from Lemma 5 that a conformal module (V, E) over a formal distribution (associative or Lie) algebra gives rise to a *module* $\mathrm{Con}(V, E) := \mathbb{C}[\partial_z]\bar{E}$ over the conformal algebra $\mathrm{Con}(A, F)$.

A conformal module (V, E) is called *finite* if $\mathrm{Con}(V, E)$ is a finitely generated $\mathbb{C}[\partial_z]$-module and is called *irreducible* if it contains no irregular submodules.

The definition of a module over a conformal algebra (associative or Lie) is motivated by (17-19) and is given along the same lines as before. A *module* over a conformal

algebra R is a $\mathbb{C}[\partial]$-module M endowed with the λ-action $a_\lambda v$ which defines a map $R \otimes_{\mathbb{C}} M \to \mathbb{C}[\lambda] \otimes_{\mathbb{C}} M$ subject to the following axioms:

$$
\begin{aligned}
(\partial a)_\lambda v &= -\lambda a_\lambda v, \quad a_\lambda \partial v = (\partial + \lambda)(a_\lambda v), && (20)\\
a_\lambda(b_\mu v) &= (a_\lambda b)_{\lambda+\mu} v \quad \text{if } R \text{ is associative}, && (21)\\
[a_\lambda, b_\mu]\, v &= [a_\lambda b]_{\lambda+\mu}\, v \quad \text{if } R \text{ is Lie}. && (22)
\end{aligned}
$$

A 1-dimensional (over $\mathbb{C}$) $\mathbb{C}[\partial]$-module over a conformal algebra R is called *trivial* if $a_\lambda = 0$ for all $a \in R$.

Remark 5.1. It is easy to see that $(b_{-\lambda-\partial} a)_\mu v = (b_{\mu-\lambda} a)_\mu v$. It follows that a module over a ($\mathbb{Z}_2$-graded) associative conformal algebra is a module over the corresponding Lie conformal (super)algebra defined by (16).

In the same way as in Section 3 we have an equivalence of categories of modules over an associative (resp. Lie) conformal algebra R and equivalence classes of conformal modules over the associative (resp. Lie) algebra Alg$R$. Proof of the following proposition is the same as that of Proposition 3.

Proposition 5 [DK]**.** *Let M be a module over a conformal algebra R. Then the torsion of R acts trivially on M and R acts trivially on the torsion of M.*

Example 5.1. Let A be an associative or Lie (super)algebra and let U be an A-module. Then, in the obvious way, $(U[t, t^{-1}], E)$ is a conformal module over the current algebra (see Example 2.1) with $E = \{u(z) = \sum_n (ut^n) z^{-n-1}\}_{u \in U}$. The associated module over the current conformal algebra Cur$A$ is $M(U) = \mathbb{C}[\partial] \otimes_{\mathbb{C}} U$ with the λ-action $a_\lambda u = a(u), \quad a \in A,\ u \in U$.

Example 5.2. For each Δ, $\alpha \in \mathbb{C}$ define the space of densities:
$F(\Delta, \alpha) = \mathbb{C}[t, t^{-1}] e^{-\alpha t} (dt)^{1-\Delta}$. This is a conformal module over Vect$\mathbb{C}^\times$ with $E = \{m(z) = \sum_n (t^n e^{-\alpha t} (dt)^{1-\Delta}) z^{-n-1}\}$. It is irreducible unless $\Delta = 0$ (in this case it has a regular submodule $d(\mathbb{C}[t, t^{-1}] e^{-\alpha t})$). The associated module over the Virasoro conformal algebra (see Example 3.2) is $M(\Delta, \alpha) = \mathbb{C}[\partial] m$ with the λ-action $L_\lambda m = (\partial + \alpha + \Delta\lambda) m$.

Theorem 5.1 [CK]**.**

(a) *Any non-trivial irreducible finite module over the Virasoro conformal algebra is isomorphic to $M(\Delta, \alpha)$ with $\Delta \neq 0$.*

(b) *Any non-trivial irreducible finite module over the current conformal algebra* Cur$\mathfrak{g}$, *where $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra, is isomorphic to $M(U)$ where U is a non-trivial irreducible finite-dimensional $\mathfrak{g}$-module.*

Corollary 5.

(a) *Any non-trivial irreducible finite conformal module over* (Vect$\mathbb{C}^\times, \{L\}$) *is isomorphic to one of the modules $F(\Delta, \alpha)$ with $\Delta \neq 0$.*

(b) *Any non-trivial irreducible finite conformal module over the current algebra $\mathfrak{g}[t, t^{-1}]$, where $\mathfrak{g}$ is a finite-dimensional semi-simple Lie algebra, is isomorphic to one of the modules $U[t, t^{-1}]$, where U is a finite-dimensional non-trivial irreducible $\mathfrak{g}$-module.*

Remark 5.2. Complete reducibility of finite modules over simple finite Lie conformal algebras breaks down. A classification of all extensions between finite irreducible modules over semisimple Lie conformal algebras is given in [CKW]. In particular, it is shown there that there exists a non-trivial extension of modules $0 \to M(\Delta', \alpha') \to E \to M(\Delta, \alpha) \to 0$ iff $\alpha = \alpha'$ and the pair (Δ, Δ') takes one of the following values (cf. [F]):
(i) $(a, a),\ (a+2, a),\ (a+3, a),\ (a+4, a)$ where $a \in \mathbb{C}$,
(ii) $(1, 0),\ (5, 0),\ (1, -4),\ ((7 \pm \sqrt{19})/2, (-5 \pm \sqrt{19})/2)$.

Remark 5.3. Theorem 5.1(b) still holds if $\mathfrak{g}$ is a finite-dimensional simple Lie superalgebra whose maximal reductive subalgebra is semisimple. However, in the remaining cases, namely $A(m, n)$ with $m \neq n$, $C(n)$ and $W(n)$, the description of finite irreducible $\mathrm{Cur}\mathfrak{g}$-modules is much more interesting (see [CK]).

The following theorem is a conformal analogue of the classical Lie and Engel theorems.

Theorem 5.2 [DK]**.** (a) *For any finite module M over a finite solvable Lie conformal algebra R there exists a non-zero vector $v \in M$ such that $a_\lambda v = c(a, \lambda)v$, $a \in R$, $c(a, \lambda) \in \mathbb{C}[\lambda]$.*

(b) *Let M be a finite module over a finite Lie conformal algebra such that the operator a_λ is nilpotent on $\mathbb{C}[\lambda] \otimes_{\mathbb{C}} M$ for each $a \in R$. Then there exists a non-zero vector in M annihilated by all operators a_λ.*

It is not difficult to prove the following analogue of the classical Burnside theorem.

Theorem 5.3. *Any non-trivial irreducible finite module over the associative conformal algebra $\mathrm{Cur}(\mathrm{Mat}_N)$ is isomorphic to $M(\mathbb{C}^N)$, where $\mathbb{C}^N$ is the standard Mat_N-module. Any finite module over $\mathrm{Cur}(\mathrm{Mat}_N)$ is a direct sum of irreducible modules.*

6 Conformal algebras $\mathrm{Cend}M$ and gcM

Let U and V be two $\mathbb{C}[\partial]$-modules. A *conformal linear map* from U to V is a $\mathbb{C}$-linear map $a : U \to \mathbb{C}[\lambda] \otimes_{\mathbb{C}} V$, denoted by $a_\lambda : U \to V$, such that $\partial a_\lambda - a_\lambda \partial = -\lambda a_\lambda$. Denote the vector space of all such maps by $\mathrm{Chom}(U, V)$. It has a canonical structure of a $\mathbb{C}[\partial]$-module: $(\partial a)_\lambda = -\lambda a_\lambda$.

Remark 6.1. If U and V are modules over a Lie conformal algebra R, then the $\mathbb{C}[\partial]$-module $\mathrm{Chom}(U, V)$ carries an R-module structure as well, defined by $(a_\lambda \varphi)_\mu u = a_\lambda(\varphi_{\mu-\lambda} u) - \varphi_{\mu-\lambda}(a_\lambda u)$, where $a \in R, u \in U,\ \varphi \in \mathrm{Chom}(U, V)$. Hence we may define the contragredient R-module: $U^* = \mathrm{Chom}(U, \mathbb{C})$, where $\mathbb{C}$ is the trivial R-module, and the tensor product of R-modules: $U \otimes V = \mathrm{Chom}(U^*, V)$.

In the special case $U = V = M$ we let $\mathrm{Cend}M = \mathrm{Chom}(M, M)$. This $\mathbb{C}[\partial]$-module has the following λ-product making it an associative conformal algebra $(a, b \in \mathrm{Cend}M)$:

$$(a_\lambda b)_\mu v = a_\lambda(b_{\mu-\lambda} v), \quad v \in M. \tag{23}$$

Indeed, (12) is immediate, while (13) follows from (23) by replacing μ by $\mu + \lambda$.

Remark 6.2. The λ-bracket $[a_\lambda b] = a_\lambda b - b_{-\lambda-\partial}a$ makes Cend$M$ a Lie conformal algebra denoted by gcM. (A decomposition of M in a direct sum of $\mathbb{C}[\partial]$-modules $M = M_{\bar{0}} \oplus M_{\bar{1}}$ induces a $\mathbb{Z}_2$-gradation on the algebra Cend$M$ and the λ-bracket (16) makes it a Lie conformal superalgebra.) Due to Remark 5.1 one has a simpler form of this λ-bracket:

$$[a_\lambda b]_\mu v = [a_\lambda, b_{\mu-\lambda}]v, \quad v \in M. \tag{24}$$

Remark 6.3. Formulas (23) (resp. (24)) show that a structure of a module over an associative (resp. Lie) conformal algebra R is the same as a homomorphism of R to the conformal algebra Cend$M$ (resp. gcM).

Let N be a positive integer, and let $\mathrm{Cend}_N = \mathrm{Cend}\mathbb{C}[\partial]^N$, $gc_N = gc\,\mathbb{C}[\partial]^N$, where $\mathbb{C}[\partial]^N$ denotes the free $\mathbb{C}[\partial]$-module of rank N. Remark 6.2 shows that the conformal algebras Cend_N and gc_N play the same role in the theory of conformal algebras as End_N and gl_N play in the theory of associative and Lie algebras. Below we give a less abstract description of these conformal algebras.

Let $\mathrm{Diff}_N\mathbb{C}^\times$ be the associative algebra of all $N \times N$-matrix valued regular differential operators on $\mathbb{C}^\times$. It is spanned over $\mathbb{C}$ by differential operators $At^j\partial_t^m$, where $A \subset \mathrm{Mat}_N$, $j \in \mathbb{Z}$, $m \in \mathbb{Z}_+$. Introduce the following formal distribution for $A \in \mathrm{Mat}_N$, $m \in \mathbb{Z}_+$: $J_A^m(z) = \sum_{j\in\mathbb{Z}} At^j(-\partial_t)^m z^{-j-1}$. Then we have:

$$J_A^m(z)J_B^n(w) = \sum_{j=0}^{m}\sum_{i=0}^{j}\binom{m}{j}\binom{j}{i}\partial_w^{j-i}J_{AB}^{m+n-j}(w)\partial_w^i\delta(z-w). \tag{25}$$

It follows that $F = \{J_A^m(z)\}_{m\in\mathbb{Z}_+,\, A\in\mathrm{Mat}_N}$ is a local family, hence $(\mathrm{Diff}_N\mathbb{C}^\times, F)$ is a formal distribution associative algebra. The corresponding associative conformal algebra is

$$\mathrm{Con}(\mathrm{Diff}_N\mathbb{C}^\times,\, F) = \sum_{\substack{m\in\mathbb{Z}_+\\ A\in\mathrm{Mat}_N}} \mathbb{C}[\partial]J_A^m$$

with the λ-product, derived from (25) to be:

$$J_A^m{}_\lambda J_B^n = \sum_{j=0}^{m}\binom{m}{j}(\lambda+\partial)^j J_{AB}^{m+n-j}. \tag{26}$$

The obvious representation of $\mathrm{Diff}_N\mathbb{C}^\times$ on the space $\mathbb{C}^N[t,t^{-1}]e^{-\alpha t}$ is an irreducible conformal module with the family $E = \{a(z) = \sum_{m\in\mathbb{Z}}(at^m e^{-\alpha t})z^{-m-1}\}_{a\in\mathbb{C}^N}$. The associated module over $\mathrm{Con}(\mathrm{Diff}_N\mathbb{C}^\times, F)$ is $\mathbb{C}[\partial]^N = \mathbb{C}[\partial]\otimes_{\mathbb{C}}\mathbb{C}^N$ with the λ-action

$$J_A^m{}_\lambda v = (\partial+\lambda+\alpha)^m Av, \quad m \in \mathbb{Z}_+,\ v \in \mathbb{C}^N. \tag{27}$$

By Remark 6.3, representation (27) gives us associative conformal algebra homomorphisms $\varphi_\alpha : \mathrm{Con}(\mathrm{Diff}_N\mathbb{C}^\times, F) \to \mathrm{Cend}_N$.

Proposition 6 [DK]. *The homomorphisms φ_α are isomorphisms.*

Proof. We have by (27): $(\partial^k J_A^m)_\lambda v = (-\lambda)^k(\lambda+\partial+\alpha)^m Av$. Since a conformal linear map is determined by its values on a set of generators of a $\mathbb{C}[\partial]$-module and the polynomials $\lambda^k(\lambda+\partial+\alpha)^m v$ with $k, m \in \mathbb{Z}_+$, $v \in \mathbb{C}^N$, span $\mathbb{C}^N[\lambda, \partial]$, the proposition follows.

The representation (27) of the associative conformal algebra Cend_N in $\mathbb{C}[\partial]^N$ can be generalized by keeping formula (27), but replacing $\mathbb{C}[\partial]^N$ by $\mathbb{C}[\partial]^N \otimes U$ and $\alpha \in \mathbb{C}$ by an indecomposable linear operator α on U. Denote this representation of Cend_N by σ_α^{as}. This representation gives rise to the representation of the associated Lie conformal algebra gc_N (see Remark 6.2), which we denote by σ_α. The representation σ_α^* contragredient to σ_α (see Remark 6.1) is again a free $\mathbb{C}[\partial]$-module with the following λ-action: $J_A^m{}_\lambda v = -(-\partial-\alpha^*)^m({}^tAv)$, for $m \in \mathbb{Z}_+$, $v \in \mathbb{C}^N \otimes U^*$.

Theorem 6.1 [BKRW]**.** *The representations σ_α and σ_α^* are all non-trivial finite indecomposable gc_N-modules.*

The proof of this theorem relies on methods developed in [KR] and [CKW]. The analogue of Theorem 6.1(a),(b) in the associative case is the following Burnside type theorem.

Theorem 6.2. *The* Cend_N*-modules σ_α^{as} are all non-trivial finite irreducible* Cend_N*-modules.*

Given a collection $P = (P_1(\partial_t), \dots, P_n(\partial_t))$ of non-zero polynomials, one has a formal distribution subalgebra of $\mathrm{Diff}_N\mathbb{C}^\times$ consisting of matrices whose i-th column is divisible by P_i, $i = 1, \dots, n$. The corresponding subalgebra of Cend_N, denoted by $\mathrm{Cend}_{N,P}$, still acts irreducibly on $\mathbb{C}[\partial]^N$.

Conjecture 6. *If $R \subset \mathrm{Cend}_N$ is a subalgebra which still acts irreducibly on $\mathbb{C}[\partial]^N$, then either R is conjugate to* $\mathrm{Cur}(\mathrm{Mat}_N\,\mathbb{C})$ *or R is one of the subalgebras* $\mathrm{Cend}_{N,P}$.

7 Γ-twisted and Γ-local formal distribution algebras [GK], [BDK]

We discuss here two generalizations of the notion of a formal distribution algebra which incorporate the examples of twisted current algebras and Ramond type superalgebras.

The first generalization requires consideration of non-integral powers of indeterminates. Let Γ be an additive subgroup of $\mathbb{C}$ containing $\mathbb{Z}$. For $\alpha \in \Gamma$ we denote by $\bar{\alpha}$ the coset $\alpha + \mathbb{Z}$.

An $(\bar{\alpha}, \bar{\beta}, \dots)$-*twisted* U-valued formal distribution is a series of the form

$$a(z, w, \dots) = \sum_{\substack{m \in \bar{\alpha} \\ n \in \bar{\beta}}} a_{m,n,\dots} z^{-m-1} w^{-n-1} \dots .$$

An $(\bar{\alpha}, \bar{\beta})$-twisted formal distribution $a(z, w)$ is called *local* if, as before, $(z-w)^N a(z,w) = 0$ for a sufficiently large N. An example of a $(\bar{\alpha}, -\bar{\alpha})$-twisted local $\mathbb{C}$-valued formal distribution is the $\bar{\alpha}$-twisted formal δ-function:

$\delta_{\bar\alpha}(z-w)=z^{-1}\sum_{n\in\bar\alpha}(w/z)^n$. As before, a $(\bar\alpha,\bar\beta)$-twisted local formal distribution $a(z,w)$ uniquely decomposes in a finite sum of the form

$$a(z,w)=\sum_{j\in\mathbb{Z}_+}c^j(w)\partial_w^{(j)}\delta_{\bar\alpha}(z-w). \tag{28}$$

Here $c^j(w)$ are $\bar\alpha+\bar\beta$-twisted formal distributions.

Let now U be an algebra. The pair consisting of an $\bar\alpha$-twisted formal distribution $a(z)$ and a $\bar\beta$-twisted distribution $b(z)$ is called a *local pair* if the $(\bar\alpha,\bar\beta)$-twisted formal distribution $a(z)b(w)$ is local. The j-th coefficient of the expansion (28) of $a(z)b(w)$ is denoted by $a(w)_{(j)}b(w)$ and is called the j-th product of these formal distributions. As before, we define the λ-product by $a(w)_\lambda b(w)=\sum_{j\in\mathbb{Z}_+}\lambda^{(j)}(a(w)_{(j)}b(w))$.

A pair (U,F), where U is an algebra and F is a family of twisted U-valued formal distributions in z, is called a Γ-*twisted formal distribution algebra* if $\bar F$ is a closed under j-th products local family of α-twisted formal distributions with $\alpha\in\Gamma$, whose coefficients span U. Note that an analogue of Lemma 2 holds. One can show that Proposition 2 holds if (U,F) is a Γ-twisted formal distribution algebra. Then $R=\mathrm{Con}(U,F):=\mathbb{C}[\partial_z]\bar F$ is a conformal algebra (which is associative, commutative or Lie if U is associative, commutative or Lie respectively). Note also that R carries a $\Gamma/\mathbb{Z}$-gradation

$$R=\oplus_{\bar\alpha\in\Gamma/\mathbb{Z}}R_{\bar\alpha}, \tag{29}$$

where $R_{\bar\alpha}$ consists of all $\bar\alpha$-twisted formal distributions from R. (It is clear that $R_{\bar\alpha(j)}R_{\bar\beta}\subset R_{\bar\alpha+\bar\beta}$ and $\partial R_{\bar\alpha}\subset R_{\bar\alpha}$.)

Thus, we get a functor Con from the category of Γ-twisted formal distribution algebras to the category of $\Gamma/\mathbb{Z}$-graded conformal algebras. Conversely, given a $\Gamma/\mathbb{Z}$-graded conformal algebra (29), we construct the corresponding Γ-twisted formal distribution algebra by letting (cf. Section 3):

$$\mathrm{Alg}R=\oplus_{\bar\alpha\in\Gamma/\mathbb{Z}}(\mathrm{Alg}R)_{\bar\alpha},$$

where $(\mathrm{Alg}R)_{\bar\alpha}$ is a quotient of the vector space with the basis $\{a_n\}_{a\in R_{\bar\alpha},\,n\in\bar\alpha}$ by the linear span of

$$\{(a+b)_n-a_n-b_n,\ (\lambda a)_n-\lambda a_n,\ (\partial a)_n+na_{n-1}\}_{a,b\in R_{\bar\alpha},\ n\in\bar\alpha}$$

with the product:

$$a_mb_n=\sum_{j\in\mathbb{Z}_+}\binom{m}{j}(a_{(j)}b)_{m+n-j},\qquad \text{where } a\in R_\alpha,\ b\in R_\beta,\ m\in\bar\alpha,\ n\in\bar\beta.$$

The local family is $F=\cup_{\bar\alpha\in\Gamma/\mathbb{Z}}\{a(z)=\sum_{n\in\bar\alpha}a_nz^{-n-1}\}_{a\in R_{\bar\alpha}}$. (Of course in the non-twisted case ($\Gamma=\{1\}$) we recover the construction of Section 3.) The relations between the functors Con and Alg are the same as in the untwisted case (see Section 3).

Example 7.1. Let $A=\oplus_{\bar\alpha\in\Gamma/\mathbb{Z}}A_{\bar\alpha}$ be a $\Gamma/\mathbb{Z}$-graded algebra. Consider the following subalgebra of the algebra $\oplus_{\alpha\in\Gamma}At^\alpha$: $U=\sum_{\alpha\in\Gamma}A_{\bar\alpha}t^\alpha$. This is a Γ-twisted formal distribution algebra with the local family $F=\cup_{\bar\alpha\in\Gamma/\mathbb{Z}}F_{\bar\alpha}$, where $F_{\bar\alpha}=\{a(z)=$

$\sum_{n\in\bar{\alpha}}(at^n)z^{-n-1}\}_{a\in A_{\bar{\alpha}}}$. (Indeed, we have: $a(z)b(w)=(ab)(w)\delta_{\bar{\alpha}}(z-w)$ for $a\in A_{\bar{\alpha}}$, $b\in A_{\bar{\beta}}$.) It is called a Γ-*twisted current algebra*. Thus, we have: $\mathrm{Con}(U,F)=\oplus_{\bar{\alpha}\in\Gamma/\mathbb{Z}}\mathbb{C}[\partial]A_{\bar{\alpha}}$ is a $\Gamma/\mathbb{Z}$-graded current conformal algebra.

Remark 7.1. The classification of gradings of R depends on the description of its automorphism group $\mathrm{Aut}R$. It is easy to see that $\mathrm{Aut}(\mathrm{Cur}A)=\mathrm{Aut}A$, provided that $aA\neq 0$ if $a\neq 0$. Indeed, applying an automorphism σ to $a_\lambda b=ab$, $a,b\in A$, we have $P(-\lambda)Q(\lambda+\partial)ab=R(\partial)ab$, where $\sigma(a)=P(\partial)a$, etc. It follows that $P(\lambda)$ is independent of λ, hence $\sigma(A)\subset A$. One can show in a similar way that the group of automorphisms of the Virasoro conformal algebra is trivial.

Remark 7.2. Let σ be an order m automorphism of a conformal algebra R. It defines a $\Gamma=\frac{1}{m}\mathbb{Z}/\mathbb{Z}$-grading of R. Let $\mathrm{Alg}(R,\sigma)$ denote the corresponding maximal Γ-twisted formal distribution algebra. One can show that $\mathrm{Alg}(R,\sigma_i)$, $i=1,2$, are isomorphic if σ_1 and σ_2 lie in the same connected component of the group $\mathrm{Aut}R$.

The second generalization deals with the usual formal distributions, but a more general notion of locality. Let Γ be a multiplicative subgroup of $\mathbb{C}^\times$. A formal distribution $a(z,w)$ is called Γ-local if $P(z/w)a(z,w)=0$ for some polynomial $P(x)$ all of whose roots lie in Γ. Obviously, for $\Gamma=\{1\}$ we have the usual locality. The special case when $P(x)$ has no multiple roots was studied in detail in [GK]. An example of a Γ-local $\mathbb{C}$-valued formal distribution is $\delta(z-\alpha w)$ where $\alpha\in\Gamma$. As before, a Γ-local U-valued formal distribution $a(z,w)$ uniquely decomposes in a finite sum of the form:

$$a(z,w)=\sum_{\substack{j\in\mathbb{Z}_+\\ \alpha\in\Gamma}}c^{j,\alpha}(w)\partial_w^{(j)}\delta(z-\alpha w), \tag{30}$$

where $c^{j,\alpha}(w)$ are some U-valued formal distributions.

Let now U be an algebra. The pair of U-valued formal distributions $a(z)$ and $b(z)$ is called Γ-*local* if the formal distribution $a(z)b(w)$ is Γ-local. The (j,α)-coefficient of the expansion (30) of $a(z)b(w)$ is denoted by $a(w)_{(j,\alpha)}b(w)$ and is called the (j,α)-th product of $a(w)$ and $b(w)$. As before, we define the (λ,α)-product ($\alpha\in\Gamma$) by $a(w)_{\lambda,\alpha}b(w)=\sum_{j\in\mathbb{Z}_+}\lambda^{(j)}(a(w)_{(j,\alpha)}b(w))$ and let $a(w)_{\lambda,\Gamma}b(w)=a(w)_{\lambda,1}b(w)$.

For $\alpha\in\Gamma$ introduce the following linear operator T_α on the space of formal distributions: $T_\alpha(a(z))=\alpha a(\alpha z)$. Then one has:

$$a(w)_{\lambda,\alpha}b(w) = (T_\alpha a(w))_{\lambda,\Gamma}b(w), \tag{31}$$

$$T_\alpha(a(w)_{\lambda,\Gamma}b(w)) = (T_\alpha a(w))_{\alpha\lambda,\Gamma}(T_\alpha b(w)). \tag{32}$$

A pair (U,F), where U is an algebra and F is a family of U-valued formal distributions in z, is called a Γ-*local formal distribution algebra* if $\bar{F}$ consists of pairwise Γ-local formal distributions whose coefficients span U. Here $\bar{F}$ denotes the minimal family containing F, invariant under all T_α ($\alpha\in\Gamma$) and closed under all (j,α)-products with $\alpha\in\Gamma$, $j\in\mathbb{Z}_+$. Again, an analogue of Lemma 2 holds. Again, one can show that Proposition 2 holds for the (λ,Γ)-product. Thus, a Γ-local formal distribution algebra (U,F) gives rise to a conformal algebra $R=\mathrm{Con}(U,F)=\mathbb{C}[\partial_z]\bar{F}$, and an action of Γ on it by

semilinear automorphisms. "Semilinear" means that we have a homomorphism $\alpha \mapsto T_\alpha$ of Γ to the group of $\mathbb{C}$-linear invertible maps of R such that: $\partial T_\alpha = \alpha T_\alpha \partial$, $T_\alpha(a_{(j)}b) = \alpha^j (T_\alpha a)_{(j)}(T_\alpha b)$. Furthermore, for each pair $a, b \in R$, one has:

$$(T_\alpha a)_{(j)} b = 0 \quad \text{for all but finitely many } \alpha \in \Gamma, j \in \mathbb{Z}_+. \tag{33}$$

Conversely, given a conformal algebra R with an action of a group $\Gamma \subset \mathbb{C}^\times$ by semilinear automorphisms, such that (33) holds we construct the corresponding Γ-local formal distribution algebra $\mathrm{Alg}(R, \Gamma)$ which as a vector space is the quotient of $R[t, t^{-1}]$ by the linear span of elements (as before a_n stands for at^n): $\{(\partial a)_n + na_{n-1},\ (T_\alpha a)_n - \alpha^{-n} a_n\}_{a \in R,\, n \in \mathbb{Z}}$, with the following product (cf. [GK] and Section 3):

$$a_m b_n = \sum_{\substack{j \in \mathbb{Z}_+ \\ \alpha \in \Gamma}} \alpha^m \binom{m}{j} ((T_\alpha a)_{(j)} b)_{m+n-j}.$$

The Γ-local family is $F = \{a(z) = \sum_{m \in \mathbb{Z}} a_m z^{-m-1}\}_{a \in R}$.

Example 7.2. (cf. [GK]) Let A be an algebra with an action of a finite group $\Gamma \subset \mathbb{C}^\times$ by automorphisms. The action of Γ on A extends to $\mathrm{Cur} A = \mathbb{C}[\partial] \otimes_{\mathbb{C}} A$ using $\partial T_\alpha = \alpha T_\alpha \partial$, $\alpha \in \Gamma$. The corresponding Γ-local formal distribution algebra is: $a(z)b(w) = \sum_{\alpha \in \Gamma} ((T_\alpha a)b)(w) \delta(z - \alpha w)$, where $a, b \in A$. It is easy to see that we get once more a Γ-twisted current algebra.

Remark 7.3. The simplest case of simple poles considered in [GK] is the case of Γ-local formal distribution algebras which correspond to conformal algebras with the trivial action of ∂ and the λ-product independent of λ, i.e. ordinary algebras (with an action of Γ). The so called sin algebra [GL], which is a q-analogue of $\mathrm{Diff}_N \mathbb{C}^\times$, is a Γ-local formal distribution algebra associated to the algebra of infinite matrices, where $\Gamma = \{q^n\}_{n \in \mathbb{Z}}$. The role of this algebra is analogous to that of Cend_N in the theory of ordinary associative conformal algebras [GK].

Remark 7.4. One defines Γ-twisted modules and Γ-conformal modules in the obvious way, and establishes equivalence to Γ-graded and Γ-equivariant conformal modules respectively.

8 Work in progress

8.1. Case of several indeterminates [BDK]**.** Let $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_n)$, $\partial = (\partial_1, \dots, \partial_n)$ and $\lambda = (\lambda_1, \dots, \lambda_n)$. A formal distribution $a(z, w)$ with values in an algebra U is called *local* if

$$(z_i - w_i)^N a(z, w) = 0, \text{ for } i = 1, \dots, n \text{ and } N \gg 0.$$

We have a finite expansion similar to (1):

$$a(z, w) = \sum_{j \in \mathbb{Z}_+^n} c^j(w) \partial_w^{(j)} \delta(z - w),$$

where

$$\partial_w^{(j)}\delta(z-w)=\prod_i \partial_{w_i}^{(j_i)}\delta(z_i-w_i).$$

All definitions and statements of Sections 1-3 extend without difficulty to the case of several indeterminates.

Example 8.1.
(a) The Lie algebra W_n of all derivations of the algebra $\mathbb{C}[x_1,x_1^{-1},\dots,x_n,x_n^{-1}]$ is spanned by pairwise local formal distributions $A^i(z)=-\delta(z-x)\partial/\partial x_i$. The associated conformal algebra (in n indeterminates) is Con$W_n=\sum_{i=1}^n\mathbb{C}[\partial]A^i$, with the λ-bracket

$$[A^i{}_\lambda A^j]=\partial_i A^j+\lambda_i A^j+\lambda_j A^i.$$

(b) The subalgebra S_n of divergence 0 derivations is a formal distribution subalgebra of W_n. The corresponding conformal algebra is
Con$S_n=\{\sum_i P_i(\partial)A^i\,|\sum_i P_i\partial_i=0\}$.
(c) The subalgebra H_n, $n=2k$, of Hamiltonian derivations is a formal distribution subalgebra of W_n. The corresponding conformal algebra is: Con$H_n=\mathbb{C}[\partial]A$ with the λ-bracket

$$[A_\lambda A]=\sum_{i=1}^k(\lambda_{k+i}\partial_i A-\lambda_i\partial_{k+i}A).$$

(d) The subalgebra K_n, $n=2k+1$, of contact derivations is not a finite over $\mathbb{C}[\partial_1,\dots,\partial_n]$ formal distribution algebra in n indeterminates.

The structure theory of conformal algebras in several indeterminates is being worked out.

Remark 8.1. The algebra K_n is finite (with one generator) over the (non-commutative) Weyl algebra in k indeterminates. This leads to a more general notion of a conformal algebra where $\mathbb{C}[\partial_1,\dots,\partial_n]$ is replaced by a bialgebra. This is yet a special case of a general notion of a Lie* algebra introduced in [BD]. The structure theory of finite Lie* algebras is being worked out in [BDK].

8.2. Cohomology [BKV]**.** Let R be a Lie conformal algebra and let M be an R-module. An *n-cochain of R with coefficients in M* is a skewsymmetric conformal anti-linear map (cf. Section 6), i.e. a $\mathbb{C}$-linear map

$$\gamma: R^{\otimes n}\longrightarrow \mathbb{C}[\lambda_1,\dots,\lambda_n]\otimes M,\quad a_1\otimes\cdots\otimes a_n\mapsto \gamma_{\lambda_1,\dots,\lambda_n}(a_1,\dots,a_n)$$

such that $\gamma_{\lambda_1,\dots,\lambda_n}(a_1,\dots,\partial a_i,\dots,a_n)=-\lambda_i\gamma_{\lambda_1,\dots,\lambda_n}(a_1,\dots,a_i,\dots,a_n)$, and γ is skewsymmetric with respect to simultaneous permutations of a_i's and λ_i's.

A differential $d\gamma$ of a cochain γ is defined by the following formula:

$$(d\gamma)_{\lambda_1,\dots,\lambda_{n+1}}(a_1,\dots,a_{n+1})=-\sum_{i=1}^{n+1}(-1)^i a_{i_{\lambda_i}}\gamma_{\lambda_1,\dots,\hat{\lambda}_i,\dots,\lambda_{n+1}}(a_1,\dots,\hat{a}_i,\dots,a_{n+1})$$

$$+\sum_{\substack{i,j=1\\ i<j}}^{n+1}(-1)^{i+j}\gamma_{\lambda_i+\lambda_j,\lambda_1,\dots,\hat{\lambda}_i,\dots,\hat{\lambda}_j,\dots,\lambda_{n+1}}([a_{i_{\lambda_i}}a_j],a_1,\dots,,\hat{a}_i,\dots,\hat{a}_j,\dots,a_{n+1}).$$

Then $d\gamma$ is again a cochain and $d^2\gamma = 0$. Thus the cochains form a complex, called the *basic complex* and denoted by $\widetilde{C}^*(R,M) = \oplus_{n\in\mathbb{Z}_+}\widetilde{C}^n(R,M)$.

Define a structure of a $\mathbb{C}[\partial]$-module on $\widetilde{C}^*(R,M)$ by

$$(\partial\gamma)_{\lambda_1,\ldots,\lambda_n} = (\partial + \lambda_1 + \ldots + \lambda_n)\gamma_{\lambda_1,\ldots,\lambda_n}\,.$$

Then ∂ commutes with d and therefore we can define the *reduced* complex

$$C^*(R,M) = \widetilde{C}^*(R,M)/\partial\widetilde{C}^*(R,M)\,.$$

We define the *cohomology* $H^*(R,M) = \oplus_{n\in\mathbb{Z}_+}H^n(R,M)$ of R with coefficients in a module M (resp. $\widetilde{H}^*(R,M)$) to be the cohomology of the reduced complex (resp. of the basic complex).

As in the case of Lie algebra cohomology, $\widetilde{H}^0(R,M) = M^R$,
$H^1(R, Chom(N,M))$ parameterizes extensions of M by a module N, $H^2(R,\mathbb{C})$ parameterizes central extensions of R, $H^2(R,M)$ parameterizes abelian extensions of R, etc.

Example 8.2.
(a) Let R be the Virasoro conformal algebra and $M = \mathbb{C}$ be its trivial module. Then $\widetilde{H}^n(R,\mathbb{C})$ is 1-dimensional for $n = 0$ or 3 and is 0 otherwise. It follows that $H^n(R,\mathbb{C})$ is 1-dimensional for $n = 0, 2$ or 3 and is 0 otherwise. This example is intimately related to the Gelfand-Fuchs cohomology (see [F]). We also have calculated $H^*(R, M(\Delta,\alpha))$.
(b) Let $R = \mathrm{Cur}\mathfrak{g}$, where $\mathfrak{g}$ is a simple finite-dimensional Lie algebra. Then $\widetilde{H}^*(R,\mathbb{C})$ is the Grassmann algebra $G = \oplus_j G_j$ on generators of degrees $2m_1+1, 2m_2+1,\ldots$, where m_i's are the exponents of $\mathfrak{g}$, i.e. it is the same as $H^*(\mathfrak{g},\mathbb{C})$. It follows that $H^*(R,\mathbb{C}) = (\oplus_j G_j)\oplus(\oplus_j G_{j-1})$. In particular $\dim H^2(R,\mathbb{C}) = 1$. We also have calculated $H^*(R, M(U))$.

In the case when R is an associative conformal algebra one can construct analogues of Hochschild and cyclic cohomology.

8.3. Open Problem. *Classify all infinite conformal subalgebra of gc_N which still acts irreducibly on $\mathbb{C}[\partial]^N$.*

Remark 8.3. All finite subalgebras of gc_N that act irreducibly on $\mathbb{C}[\partial]^N$ are described in [DK].

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